Lecture 18

Wednesday, June 8, 2022 12:13 PM

* Prayer

* Spiritual thought

Recall line integral:

- of a scalar function: $\int_C f\,ds = \int_a^b f(r(t))\,|r'(t)|\,dt$ — find area, find mass, ...

- of a vector field: $\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$ — find work, find circulation, ...

$$\int_C \underbrace{(P,Q)}_{F}\cdot\underbrace{(dx,dy)}_{dr} = \int_C P\,dx + Q\,dy = \int_a^b P(x(t),y(t))\,x'(t)\,dt$$

$$+ \int_a^b Q(x(t),y(t))\,y'(t)\,dt$$

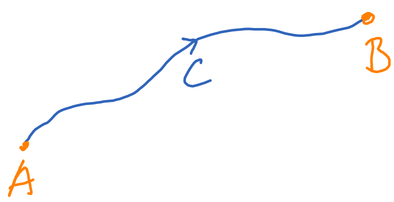
Fundamental theorem of Multivariable Calculus:

Calc I: $\int_a^b f'(x)\,dx = f(b) - f(a)$

Mult. Calc:

$$\int_a^b f_x(x,y)\,dx = f(b,y) - f(a,y)$$

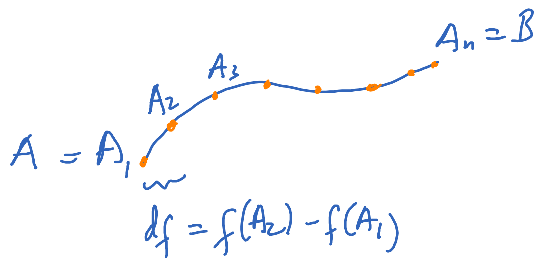
$$\int_C f_x(x,y)\,ds \neq f(B) - f(A)$$



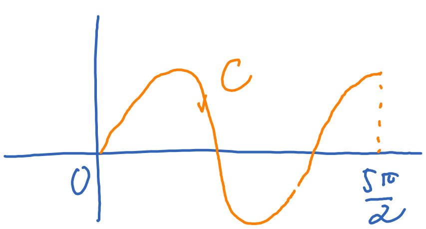
$$\int_C \nabla f \cdot dr = f(B) - f(A)$$

Why so?

$$\int_C \nabla f \cdot dr = \int_C f_x\,dx + f_y\,dy = \int_C df = f(B) - f(A)$$



Ex $\quad F(x,y) = (y, x)$ $\qquad$ Find $\int_C F \cdot dr$



C: $r(t) = (t, \sin t)$, $\quad 0 \le t \le \frac{5\pi}{2}$

*Method 1:

$$\int_C F \cdot dr = \int_0^{5\pi/2} (y, x) \cdot (1, \cos t)\,dt = \int_0^{5\pi/2} (\sin t + t\cos t)\,dt = \ldots$$

* Method 2:

$$F = (y, x) = \nabla f \quad \text{where } f(x,y) = xy$$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(B) - f(A) = f\left(\frac{5\pi}{2}, 1\right) - f(0,0) = \frac{5\pi}{2}.$$

A vector field F is said to be <u>conservative</u> if it is the gradient of some function: $F = \nabla f$. The function f is called the <u>potential function</u> of F.

Line integral of a conservative vector field is simple. But not all vector fields are conservative.

<u>Ex</u>: $F(x,y) = (\underbrace{y}_{f_x}, \underbrace{-x}_{f_y}) \rightsquigarrow f_{xy} = 1, \; f_{yx} = -1$ (!!)

How to check if a vector field is conservative in $\mathbb{R}^2$?

$$F = (P, Q)$$

Check if $P_y = Q_x$. If that is the case, the integrate P wrt x, then differentiate that wrt y, then compare with Q.

<u>Ex</u> $F(x,y) = (\underbrace{2xy + 3x^2}_{P}, \underbrace{x^2 + 2y + 1}_{Q})$.

$$\left.\begin{array}{l} P_y = 2x \\ Q_x = 2x \end{array}\right\} P_y = Q_x$$

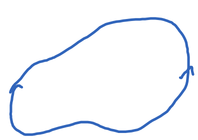
Find potential function:

$$\begin{cases} f_x = P = 2xy + 3x^2 \rightsquigarrow f = x^2y + x^3 + C(y) \to f_y = x^2 + C'(y) \\ f_y = Q = x^2 + 2y + 1 \end{cases}$$

$$\rightsquigarrow C'(y) = 2y + 1 \rightsquigarrow C(y) = y^2 + y$$

Thus, $f(x,y) = x^2y + x^3 + y^2 + y$.

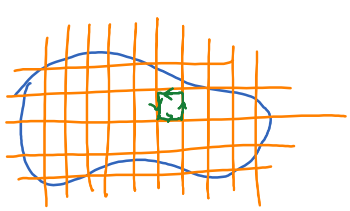
Green's theorem



$\underbrace{\int_C F \cdot dr}$ where C is a closed curve.

$=$ total circulation of F along C

If $\int_C F \cdot dr > 0$ then F, overall, flows in the orientation of C.



Partition the domain enclosed by C as follows.

In cell dA, the circulation is $\underbrace{f(x,y)}_{\text{circulation density}} dA$.

$$\text{Total circulation} = \iint_D f(x,y)\, dA$$

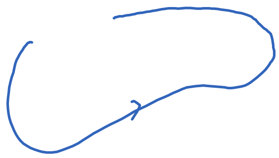
Circulation along the common edges of adjacent cells is cancelled. Thus,

$$\iint_D f(x,y)\,dA = \int_C F\cdot dr$$

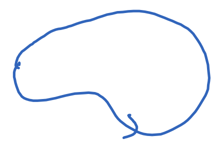
Thm (George Green 1828)

If C is a simple closed curve, positively oriented then

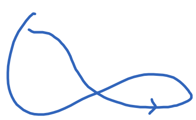
$$\int_C F\cdot dr = \iint_D (\underbrace{Q_x - P_y}_{\text{circulation density}})\,dA$$



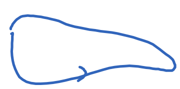
not closed



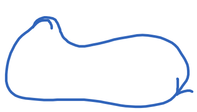
closed



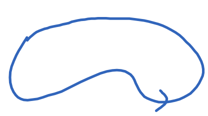
not simple



simple

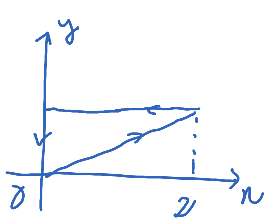


negatively oriented



positively oriented

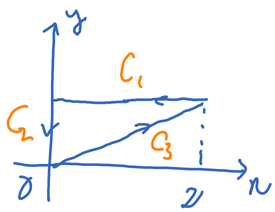
Ex



$$\int_C x\,dx + y^2\,dy = ?$$

* Method 1: use parametrization

$$\int_C x\,dx + xy\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$$



$$\int_{C_1} x\,dx + xy\,\underbrace{dy}_{0} = \int_0^2 (2-t)(-1)\,dt = \cdots$$

$(x = 2-t,\ 0 \le t \le 2)$

* Method 2: using Green's theorem

$$\int_C \underbrace{x}_{P}\,dx + \underbrace{xy}_{Q}\,dy = \iint_D (Q_x - P_y)\,dA = \iint_D y\,dA = \int_0^1 \int_0^{2y} y\,dx\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}$$